CS103
Winter 2025

Lecture 04:

First-Order Logic

Part 1 of 2

Recap from Last Time

Recap So Far

- A ***propositional variable*** is a variable that is either true or false.
- The ***propositional connectives*** are as follows:

$\rightarrow \quad \wedge \quad \top \quad \neg \quad \vee \quad \bot \quad \leftrightarrow$

p	q	$p \rightarrow q$	$p \wedge \neg q$
F	F	T	F
F	T	T	F
T	F	F	T
T	T	T	F

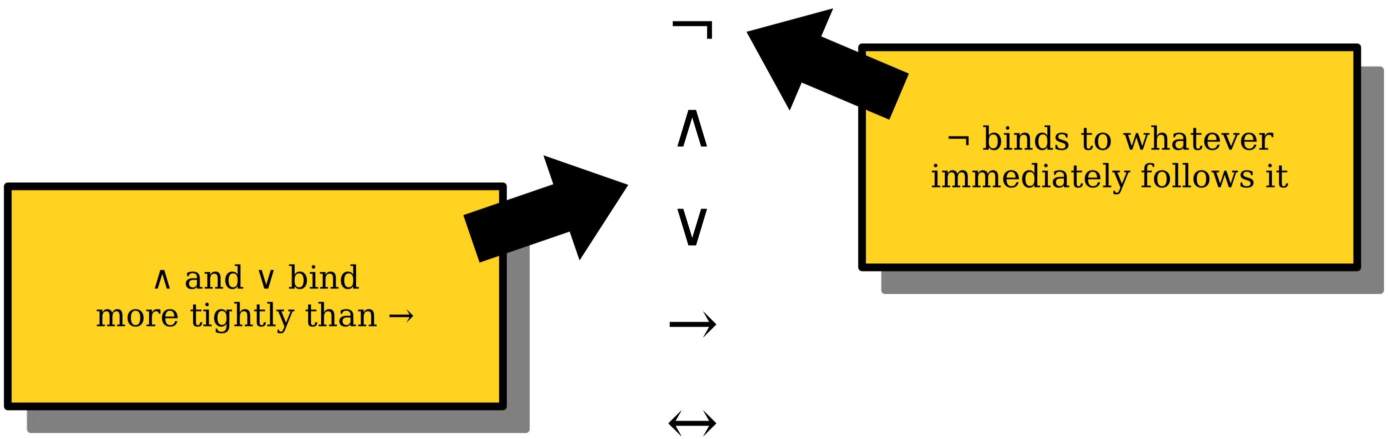
Negation of $p \rightarrow q$

Operator Precedence

- How do we parse this statement?

$$\neg x \to y \lor z \to x \lor y \land z$$

- Operator precedence for propositional logic:



- All operators are right-associative.
- We can use parentheses to disambiguate.

Why All This Matters

- Suppose we want to prove the following statement:

“If $x + y = 16$, then $x \geq 8$ or $y \geq 8$”

Theorem: If $x + y = 16$, then $x \geq 8$ or $y \geq 8$.

Proof: We will prove the contrapositive, namely, that if $x < 8$ and $y < 8$, then $x + y \neq 16$.

Pick x and y where $x < 8$ and $y < 8$. We want to show that $x + y \neq 16$. To see this, note that

$$\begin{aligned} x + y &< 8 + y \\ &< 8 + 8 \\ &= 16. \end{aligned}$$

This means that $x + y < 16$, so $x + y \neq 16$, which is what we needed to show. ■

New Stuff!

First-Order Logic

What is First-Order Logic?

- ***First-order logic*** is a logical system for reasoning about properties of objects.
- Augments the logical connectives from propositional logic with
    - ***predicates*** that describe properties of objects,
    - ***functions*** that map objects to one another, and
    - ***quantifiers*** that allow us to reason about multiple objects.

Some Examples

$$Likes(You, Eggs) \land Likes(You, Tomato) \rightarrow Likes(You, Shakshuka)$$

$$Learns(You, History) \lor ForeverRepeats(You, History)$$

$$In(MyHeart, Havana) \land TookBackTo(Him, Me, EastAtlanta)$$

$$Likes(You, Eggs) \land Likes(You, Tomato) \rightarrow Likes(You, Shakshuka)$$

$$Learns(You, History) \lor ForeverRepeats(You, History)$$

$$In(MyHeart, Havana) \land TookBackTo(Him, Me, EastAtlanta)$$

These blue terms are called *constant symbols*. Unlike propositional variables, they refer to *objects*, not propositions.

$$Likes(You, Eggs) \wedge Likes(You, Tomato) \rightarrow Likes(You, Shakshuka)$$

$$Learns(You, History) \vee ForeverRepeats(You, History)$$

$$In(MyHeart, Havana) \wedge TookBackTo(Him, Me, EastAtlanta)$$

The red things that look like function calls are called *predicates*. Predicates take objects as arguments and evaluate to true or false.

$$Likes(You, Eggs) \land Likes(You, Tomato) \rightarrow Likes(You, Shakshuka)$$

$$Learns(You, History) \lor ForeverRepeats(You, History)$$

$$In(MyHeart, Havana) \land TookBackTo(Him, Me, EastAtlanta)$$

What remains are traditional propositional connectives. Because each predicate evaluates to true or false, we can connect the truth values of predicates using normal propositional connectives.

Reasoning about Objects

- To reason about objects, first-order logic uses ***predicates***.
- Examples:

$$Cute(Quokka)$$

$$ArgueIncessantly(Democrats, Republicans)$$

- Applying a predicate to arguments produces a proposition, which is either true or false.
- Typically, when you're working in FOL, you'll have a list of predicates, what they stand for, and how many arguments they take. It'll be given separately than the formulas you write.

First-Order Formulas

- Formulas in first-order logic can be constructed from predicates applied to objects:

$$Cute(a) \to Quokka(a) \lor Kitty(a) \lor Puppy(a)$$

$$Succeeds(You) \leftrightarrow Practices(You)$$

$$x < 8 \to x < 137$$

The less-than sign is just another predicate. Binary predicates are sometimes written in *infix notation* this way.

Numbers are not "built in" to first-order logic. They're constant symbols just like "You" and "a" above.

Equality

- First-order logic is equipped with a special predicate **=** that says whether two objects are equal to one another.
- Equality is a part of first-order logic, just as $\to$ and $\neg$ are.
- Examples:

$$TomMarvoloRiddle = LordVoldemort$$

$$MorningStar = EveningStar$$

- Equality can only be applied to ***objects***; to state that two ***propositions*** are equal, use $\leftrightarrow$.

Let's see some more examples.

$$FavoriteMovieOf(You) \neq FavoriteMovieOf(Date) \land StarOf(FavoriteMovieOf(You)) = StarOf(FavoriteMovieOf(Date))$$

These purple terms are *functions*. Functions take objects as input and produce objects as output.

$$FavoriteMovieOf(You) \neq FavoriteMovieOf(Date) \land$$
$$StarOf(FavoriteMovieOf(You)) = StarOf(FavoriteMovieOf(Date))$$

Functions

- First-order logic allows ***functions*** that return objects associated with other objects.
- Examples:

$$ColorOf(Money)$$

$$MedianOf(x, y, z)$$

$$x + y$$

- As with predicates, functions can take in any number of arguments, but always return a single value.
- Functions evaluate to ***objects***, not ***propositions***.

Objects and Propositions

- When working in first-order logic, be careful to keep objects (actual things) and propositions (true or false) separate.
- You cannot apply connectives to objects:

$$\mathit{Venus} \to \mathit{TheSun}$$

- You cannot apply functions to propositions:

$$\mathit{StarOf}(\mathit{IsRed}(\mathit{Sun}) \land \mathit{IsGreen}(\mathit{Mars}))$$

- Ever get confused? *Just ask!*

The Type-Checking Table

	… operate on …	… and produce
Connectives (↔, ∧, etc.) …	propositions	a proposition
Predicates (=, etc.) …	objects	a proposition
Functions …	objects	an object

One last (and major) change

Some bear is curious.

$$\exists b.\ (Bear(b) \land Curious(b))$$

∃ is the ***existential quantifier*** and says "there is a choice of b where the following is true."

The Existential Quantifier

- A statement of the form

  $$\exists x.\ \textbf{\textit{some-formula}}$$

  is true when there exists a choice object where ***some-formula*** is true when that object is plugged in for x.

- Examples:

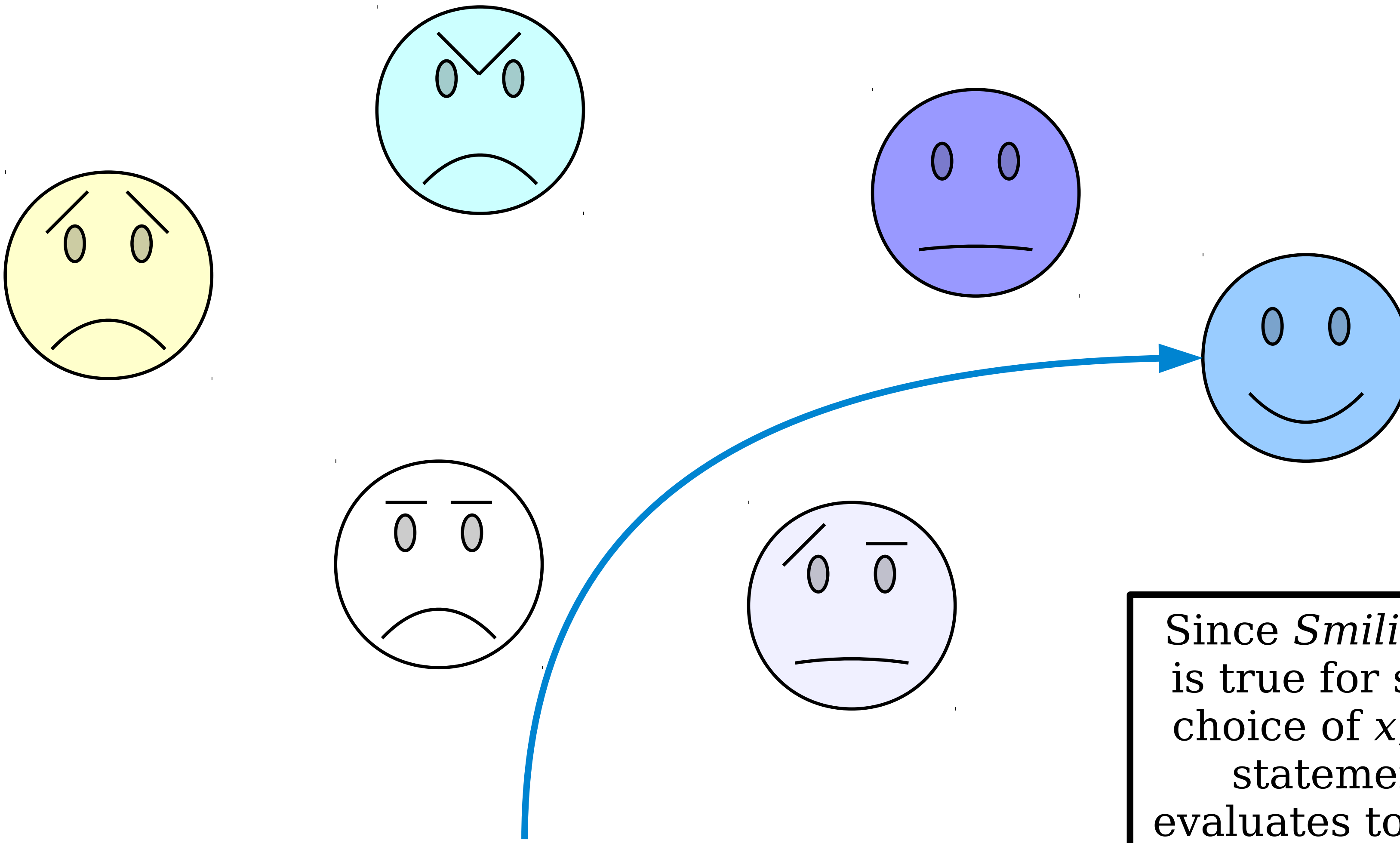
  $\exists x.\ (Even(x) \land Prime(x))$

  $\exists x.\ (TallerThan(x, me) \land WeighsLessThan(x, me))$

  $(\exists w.\ Will(w)) \to (\exists x.\ Way(x))$

- Note the two ways of applying the ∃!

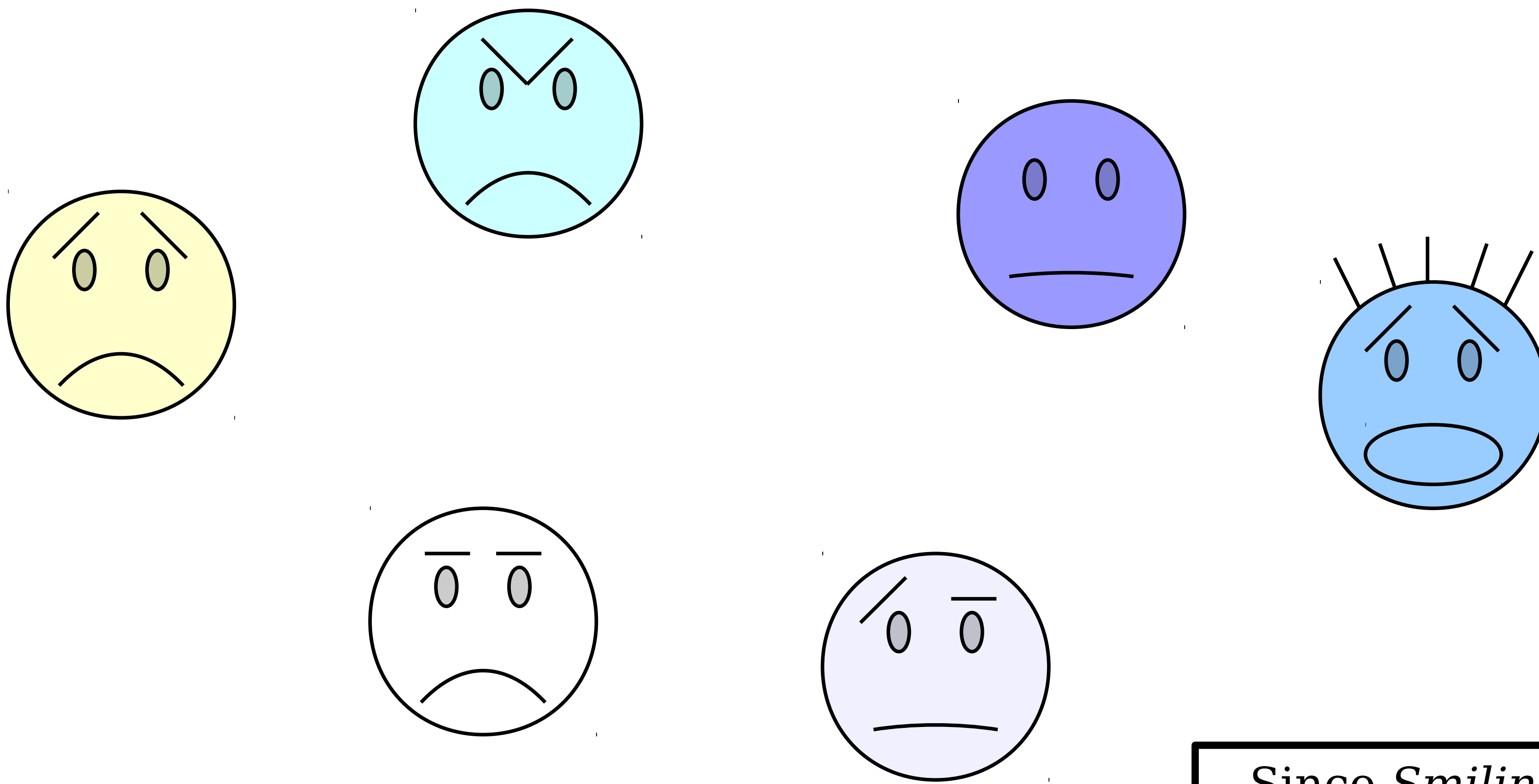
The Existential Quantifier



$\exists x.\ Smiling(x)$

Since *Smiling*(x) is true for some choice of x, this statement evaluates to true.

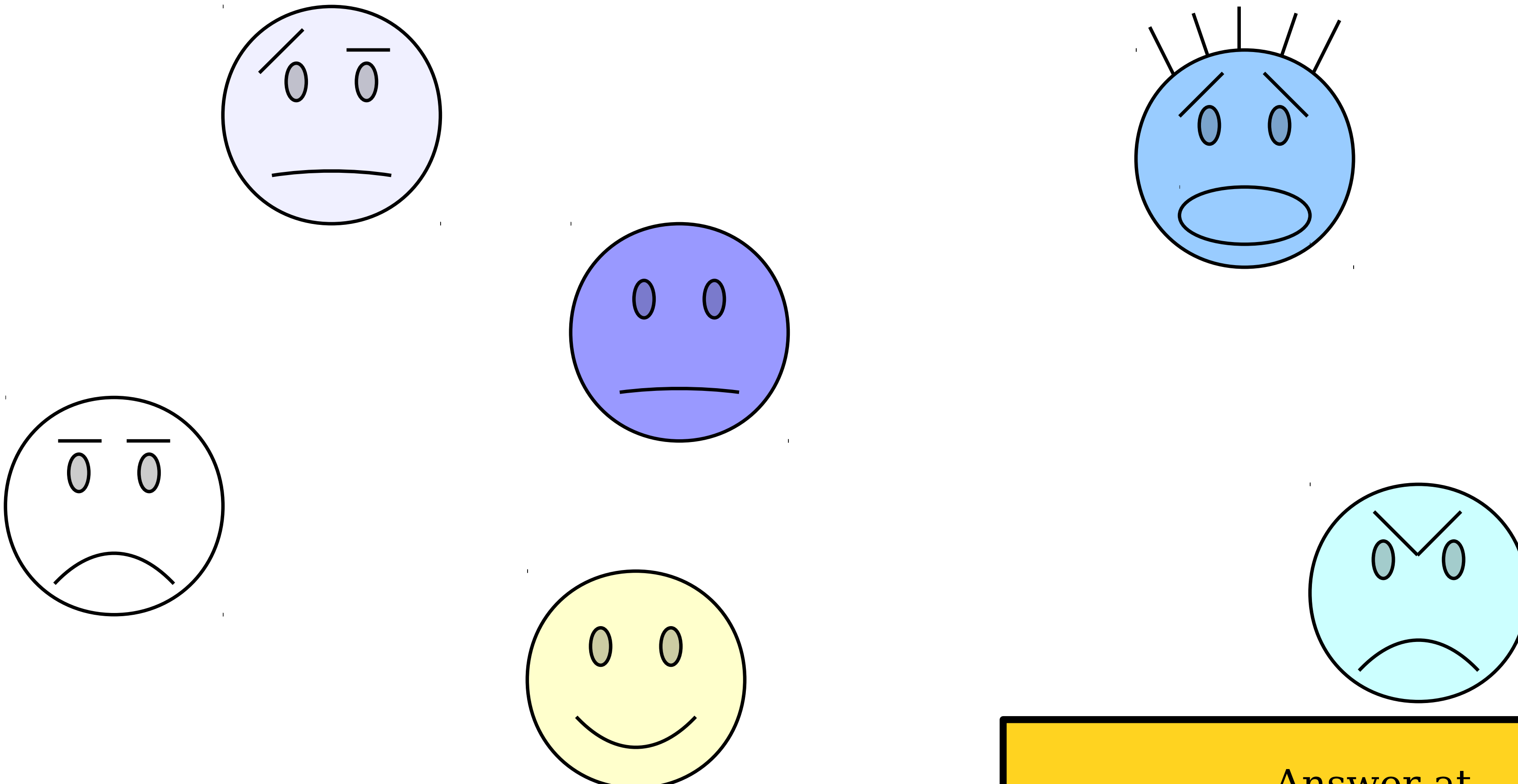
The Existential Quantifier



~~$\exists x.\ \mathit{Smiling}(x)$~~

Since $\mathit{Smiling}(x)$ is not true for any choice of x, this statement evaluates to false.

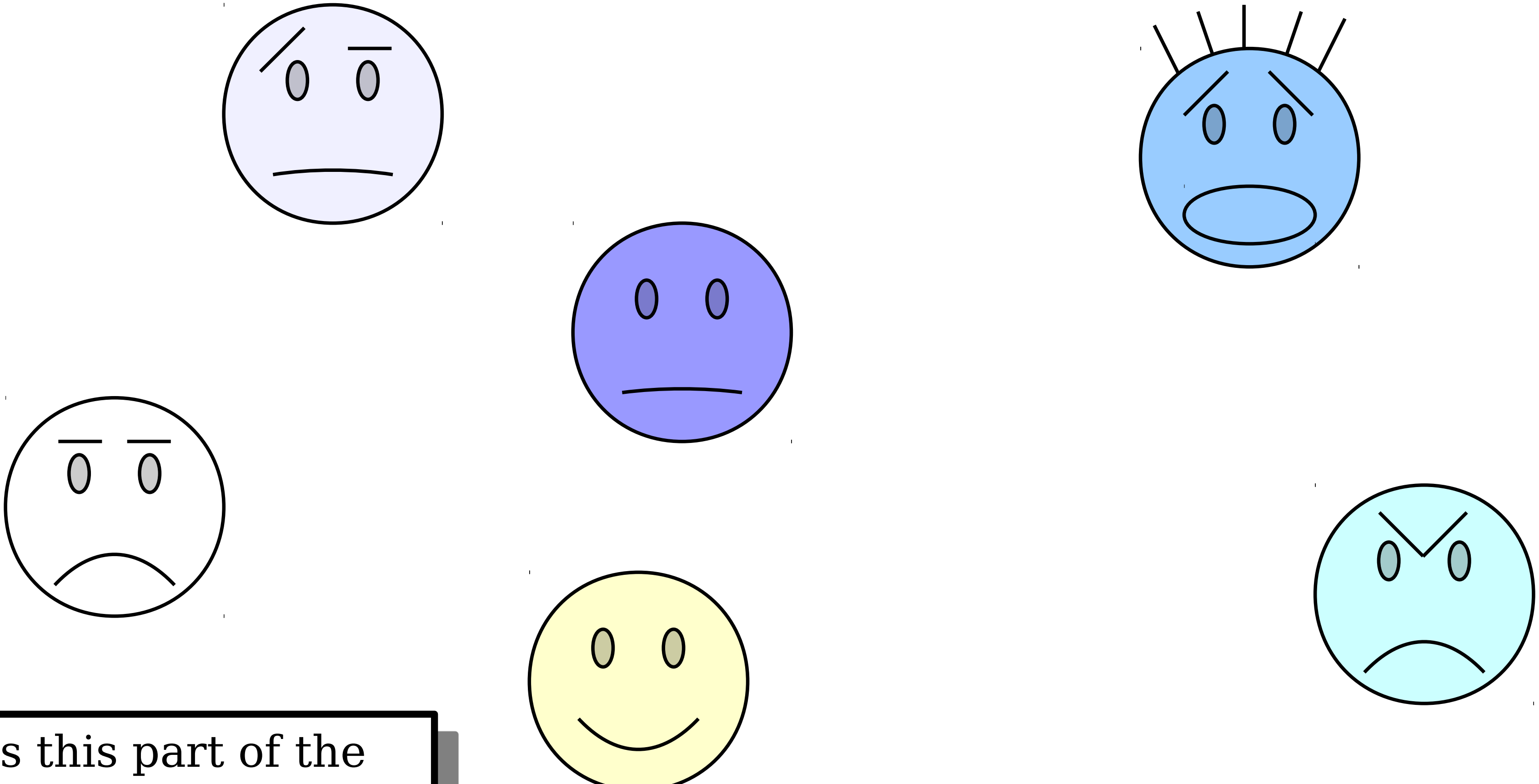
The Existential Quantifier



Answer at
https://cs103.stanford.edu/pollev

$$(\exists x.\ \mathit{Smiling}(x)) \to (\exists y.\ \mathit{WearingHat}(y))$$

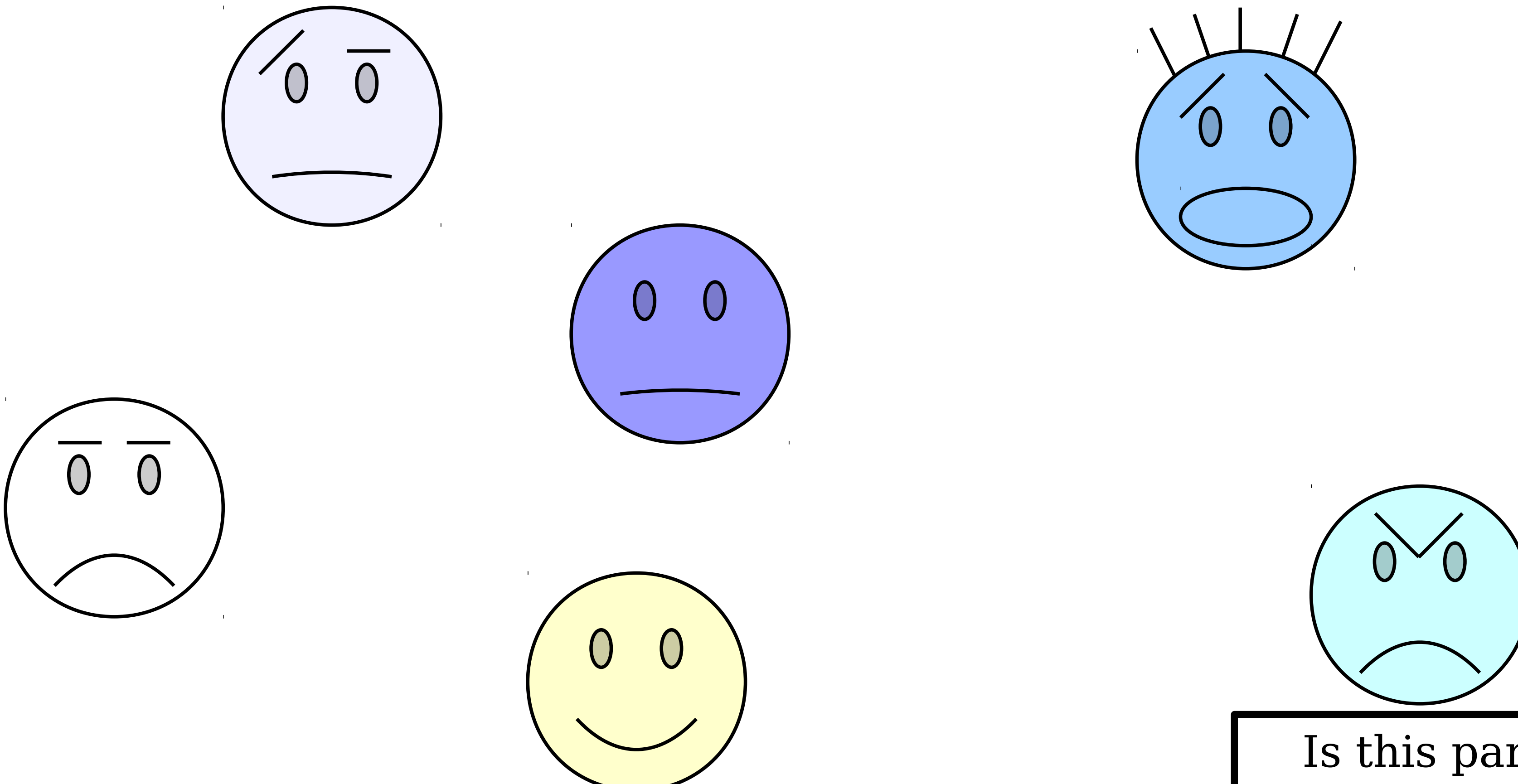
The Existential Quantifier



Is this part of the statement true or false?

$(\exists x.\ Smiling(x)) \rightarrow (\exists y.\ WearingHat(y))$

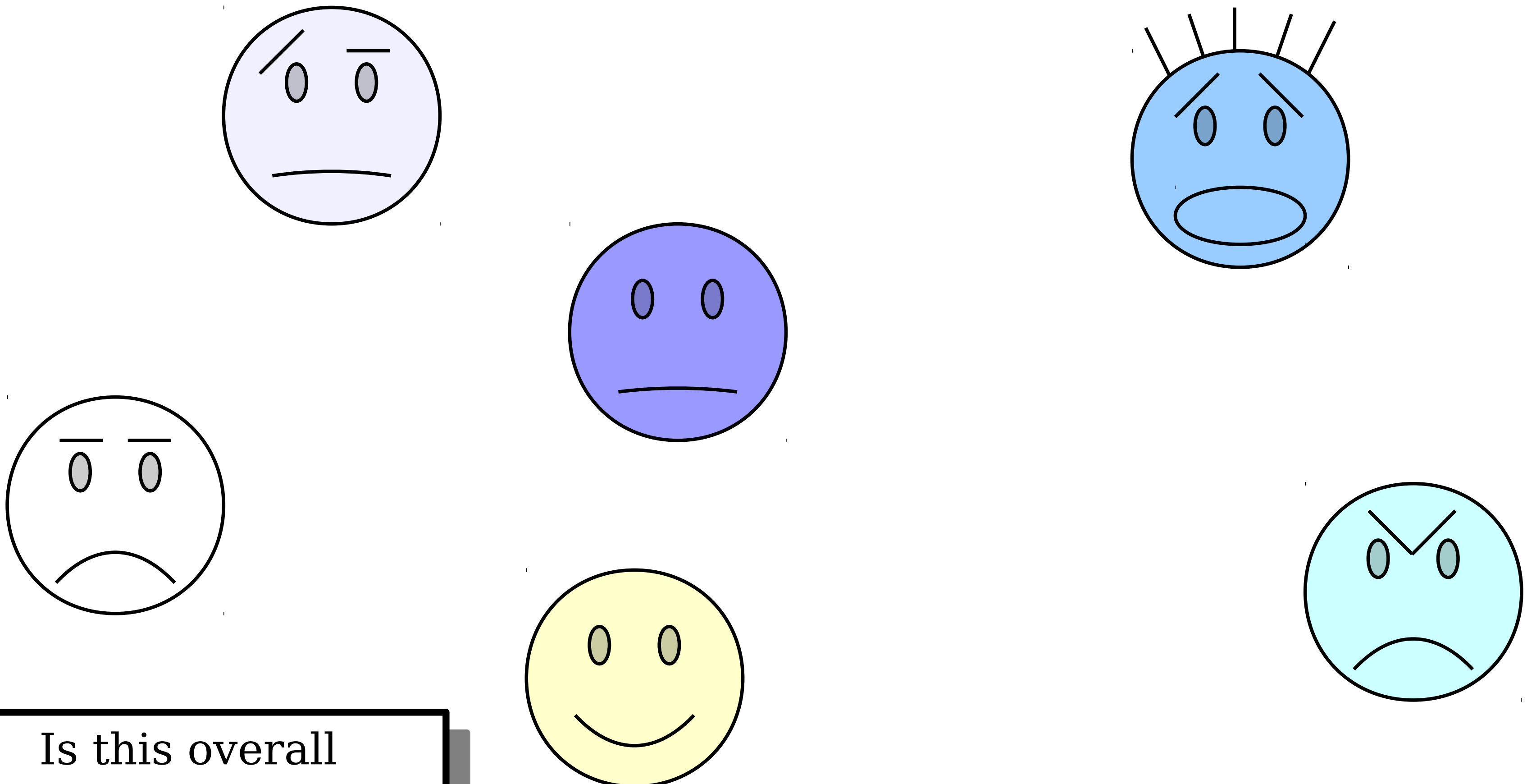
The Existential Quantifier



Is this part of the statement true or false?

$(\exists x.\ \mathit{Smiling}(x)) \rightarrow$ ~~$(\exists y.\ \mathit{WearingHat}(y))$~~

The Existential Quantifier



Is this overall statement true or false?

~~$(\exists x.\ Smiling(x)) \rightarrow (\exists y.\ WearingHat(y))$~~

Fun with Edge Cases

Existentially-quantified statements are false in an empty world, since nothing exists, period!

~~$\exists x.\ Smiling(x)$~~

Some Technical Details

Variables and Quantifiers

- Each quantifier has two parts:
  - the variable that is introduced, and
  - the statement that's being quantified.
- The variable introduced is scoped just to the statement being quantified.

$$(\exists x.\ \mathit{Loves}(\mathit{You}, x)) \land (\exists y.\ \mathit{Loves}(y, \mathit{You}))$$

The variable x just lives here.

The variable y just lives here.

Variables and Quantifiers

- Each quantifier has two parts:
  - the variable that is introduced, and
  - the statement that's being quantified.
- The variable introduced is scoped just to the statement being quantified.

$$(\exists x.\ \mathit{Loves}(\mathit{You},\ x)) \land (\exists x.\ \mathit{Loves}(x,\ \mathit{You}))$$

The variable x just lives here.

A different variable, also named x, just lives here.

Operator Precedence (Again)

- When writing out a formula in first-order logic, quantifiers have precedence just below $\neg$.
- The statement

$$\exists x.\ P(x) \land R(x) \land Q(x)$$

  is parsed like this:

$$(\exists x.\ P(x)) \land (R(x) \land Q(x))$$

- This is syntactically invalid because the variable x is out of scope in the back half of the formula.
- To ensure that x is properly quantified, explicitly put parentheses around the region you want to quantify:

$$\exists x.\ (P(x) \land R(x) \land Q(x))$$

"For any natural number *n*,
n is even if and only if n^2 is even"

$$\forall n.\ (n \in \mathbb{N} \to (Even(n) \leftrightarrow Even(n^2)))$$

∀ is the ***universal quantifier*** and says "for all choices of *n*, the following is true."

The Universal Quantifier

- A statement of the form

  $$\forall x.\ \textit{some-formula}$$

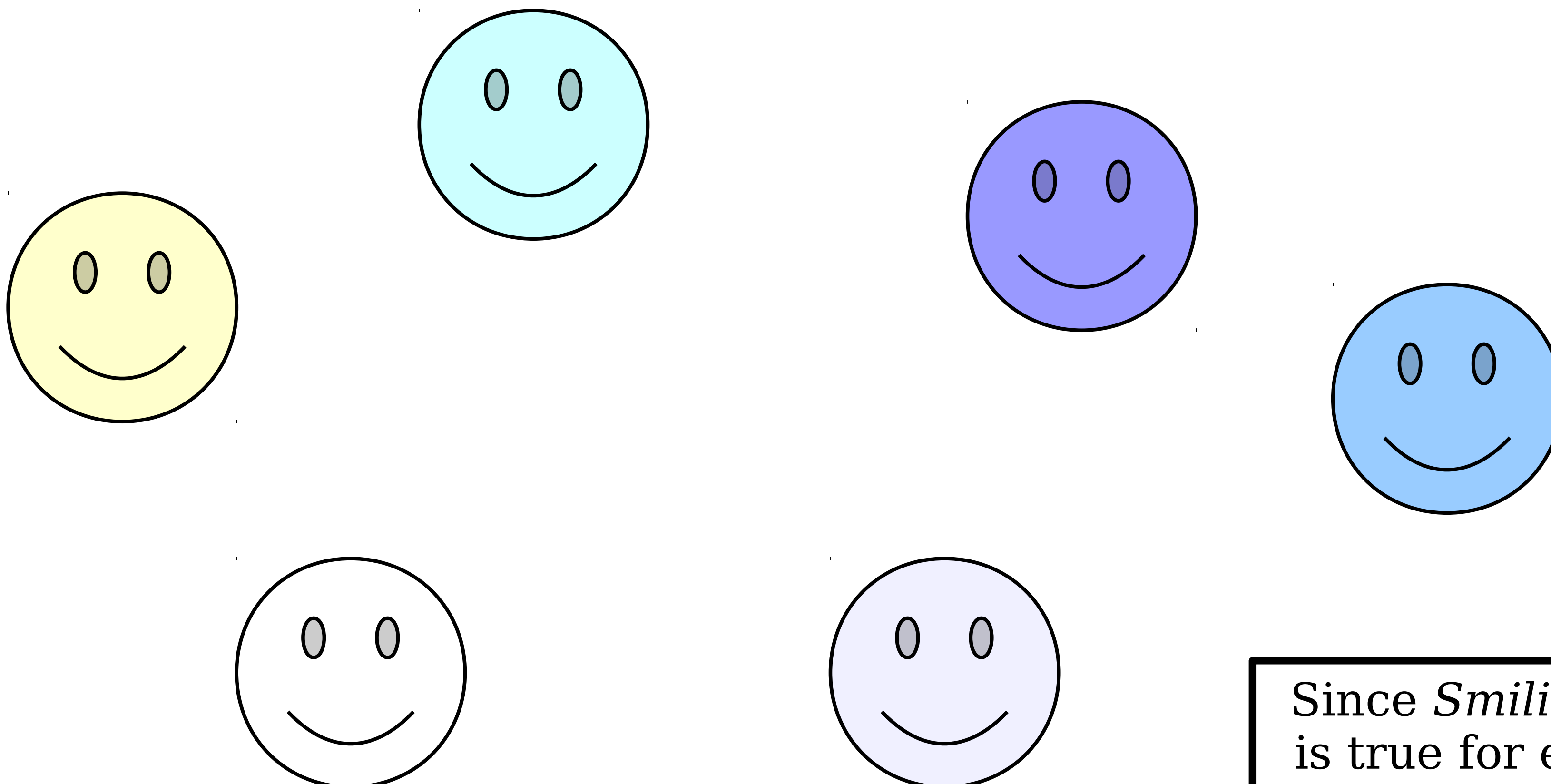
  is true when, for every choice of x, the statement ***some-formula*** is true when x is plugged into it.
- Examples:

  $$\forall p.\ (Puppy(p) \to Cute(p))$$

  $$\forall a.\ (EatsPlants(a) \lor EatsAnimals(a))$$

  $$Tallest(SultanKösen) \to \forall x.\ (SultanKösen \neq x \to ShorterThan(x, SultanKösen))$$

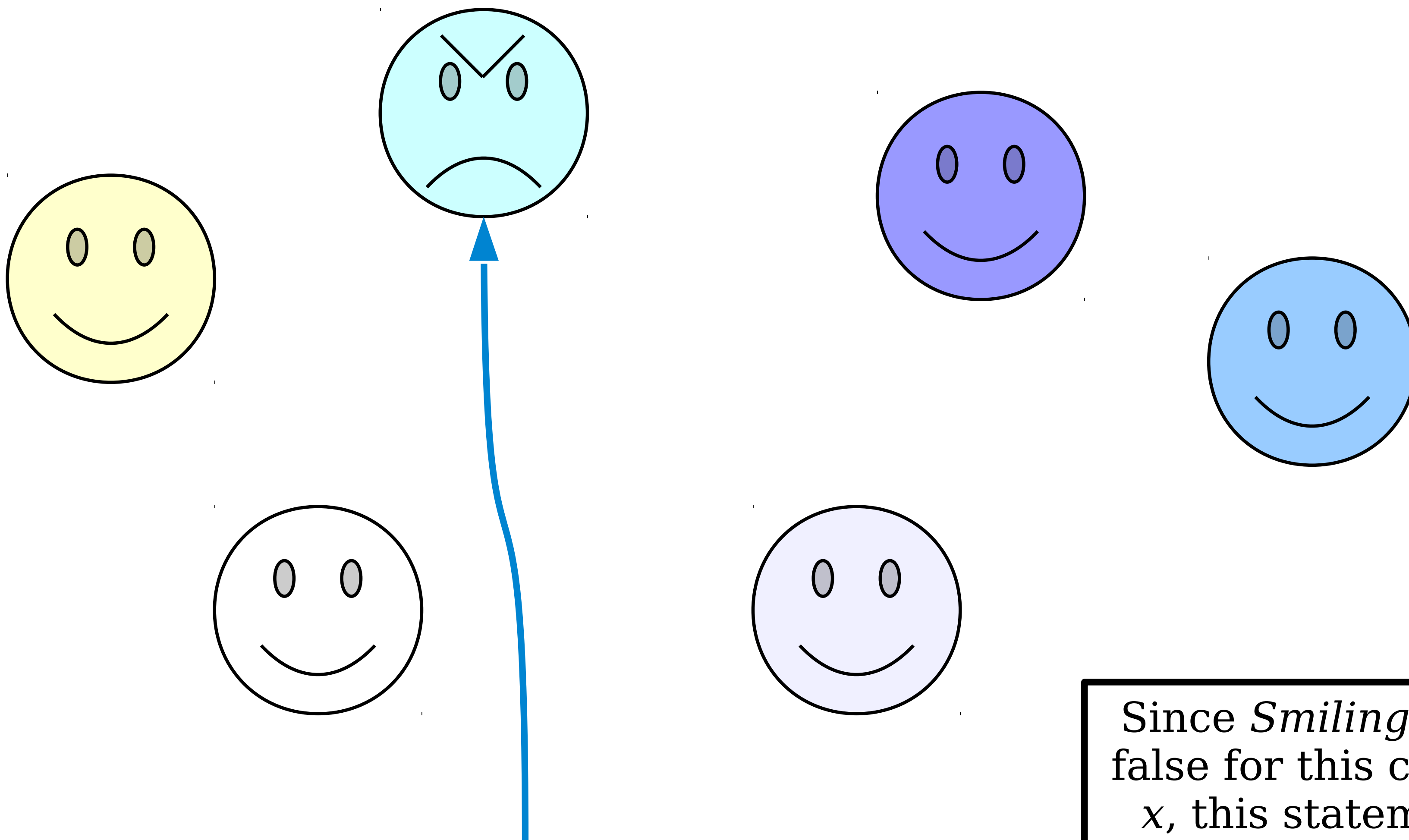
The Universal Quantifier



Since *Smiling*(x) is true for every choice of x, this statement evaluates to true.

$\forall x.\ Smiling(x)$

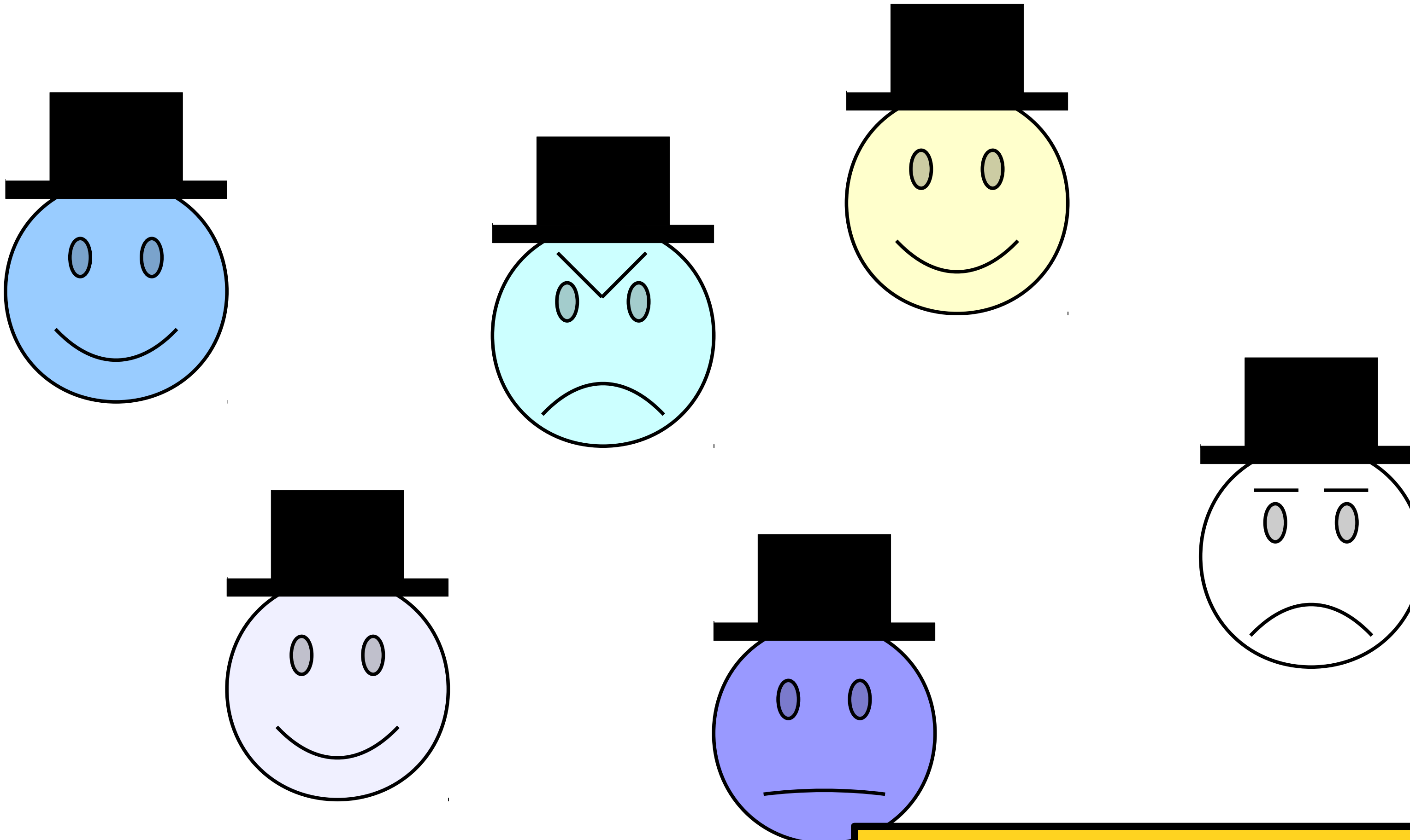
The Universal Quantifier



~~$\forall x.\ Smiling(x)$~~

Since $Smiling(x)$ is false for this choice x, this statement evaluates to false.

The Universal Quantifier



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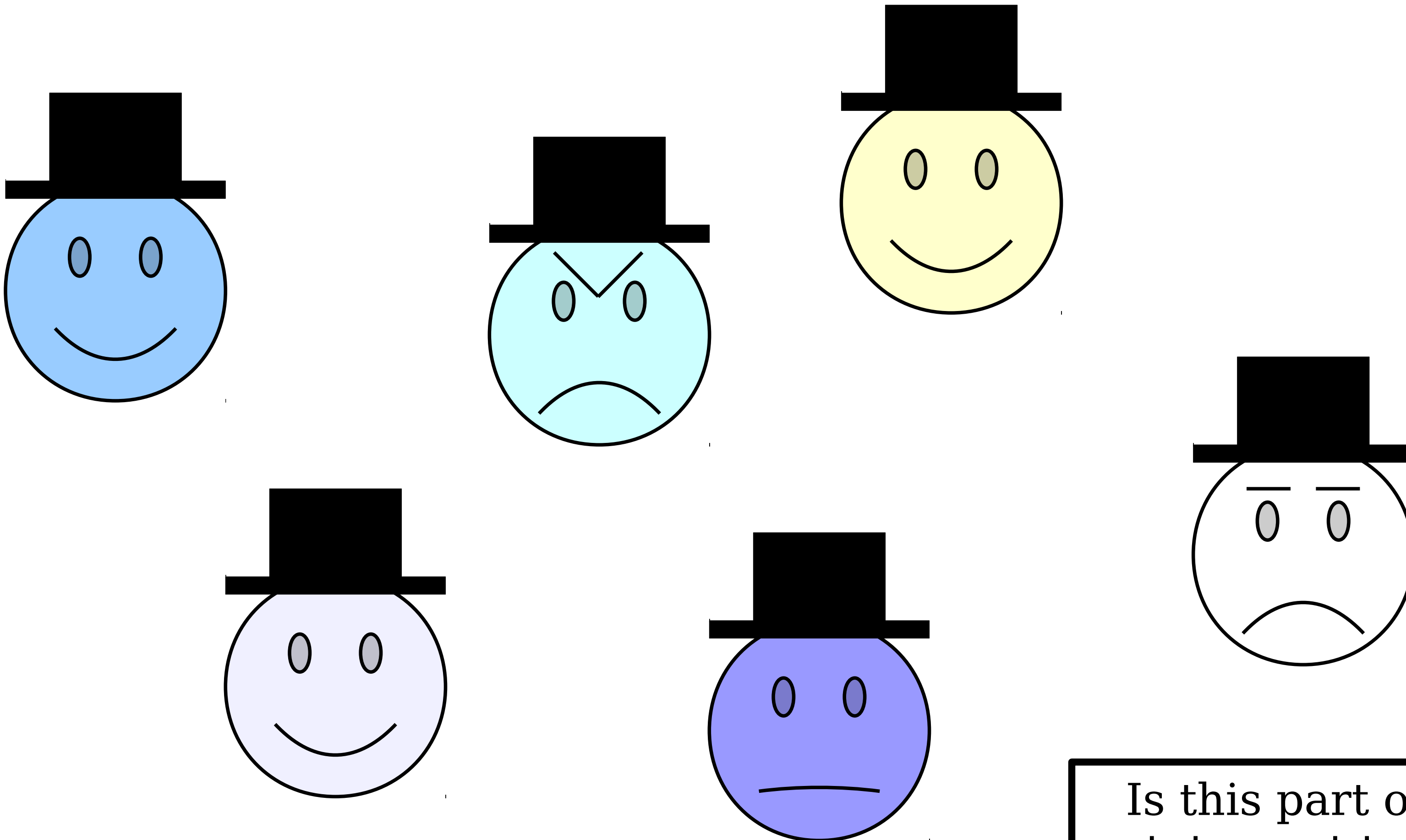
$$(\forall x.\ Smiling(x)) \rightarrow (\forall y.\ WearingHat(y))$$

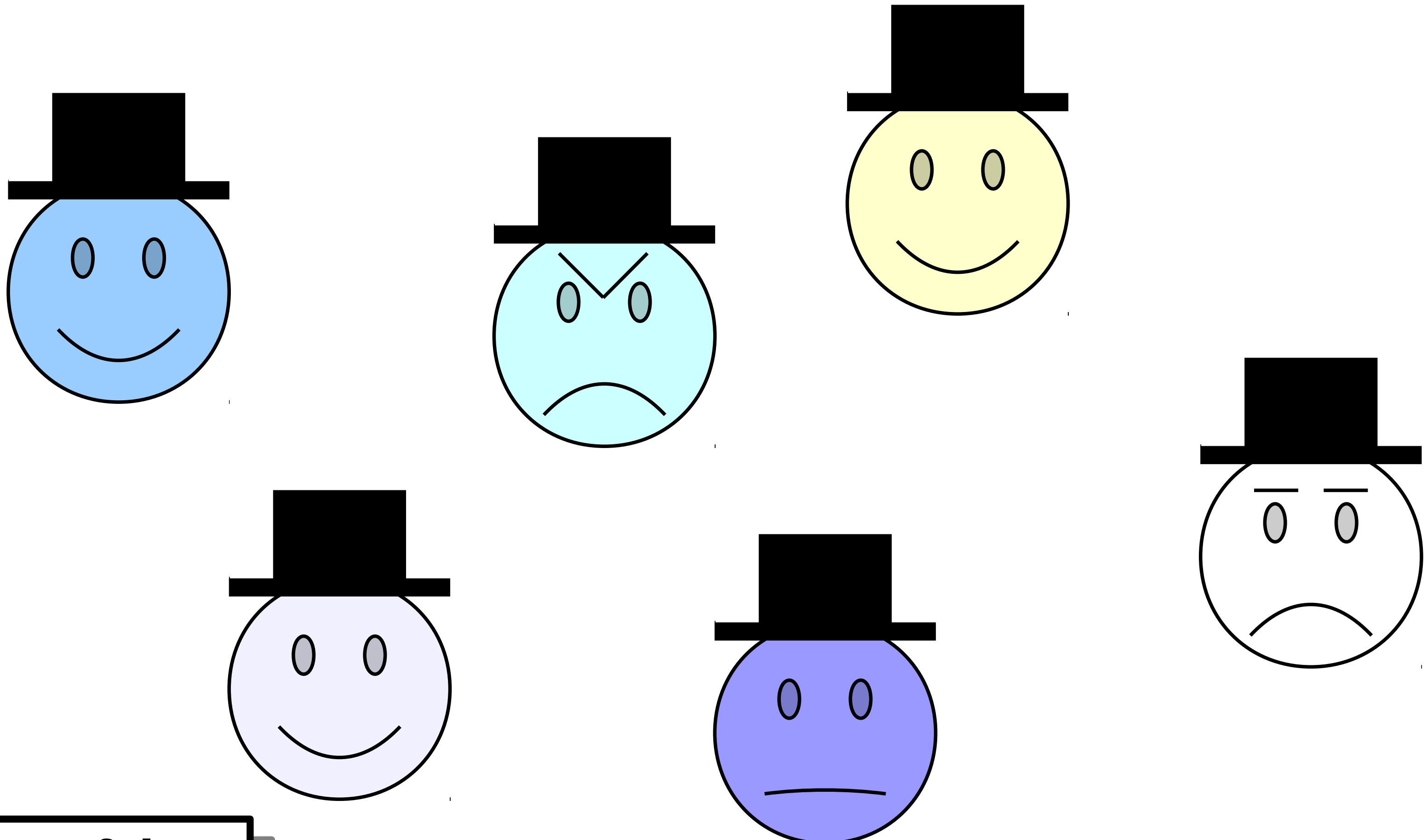
The Universal Quantifier



Is this part of the statement true or false?

$(\forall x.\ Smiling(x)) \to (\forall y.\ WearingHat(y))$

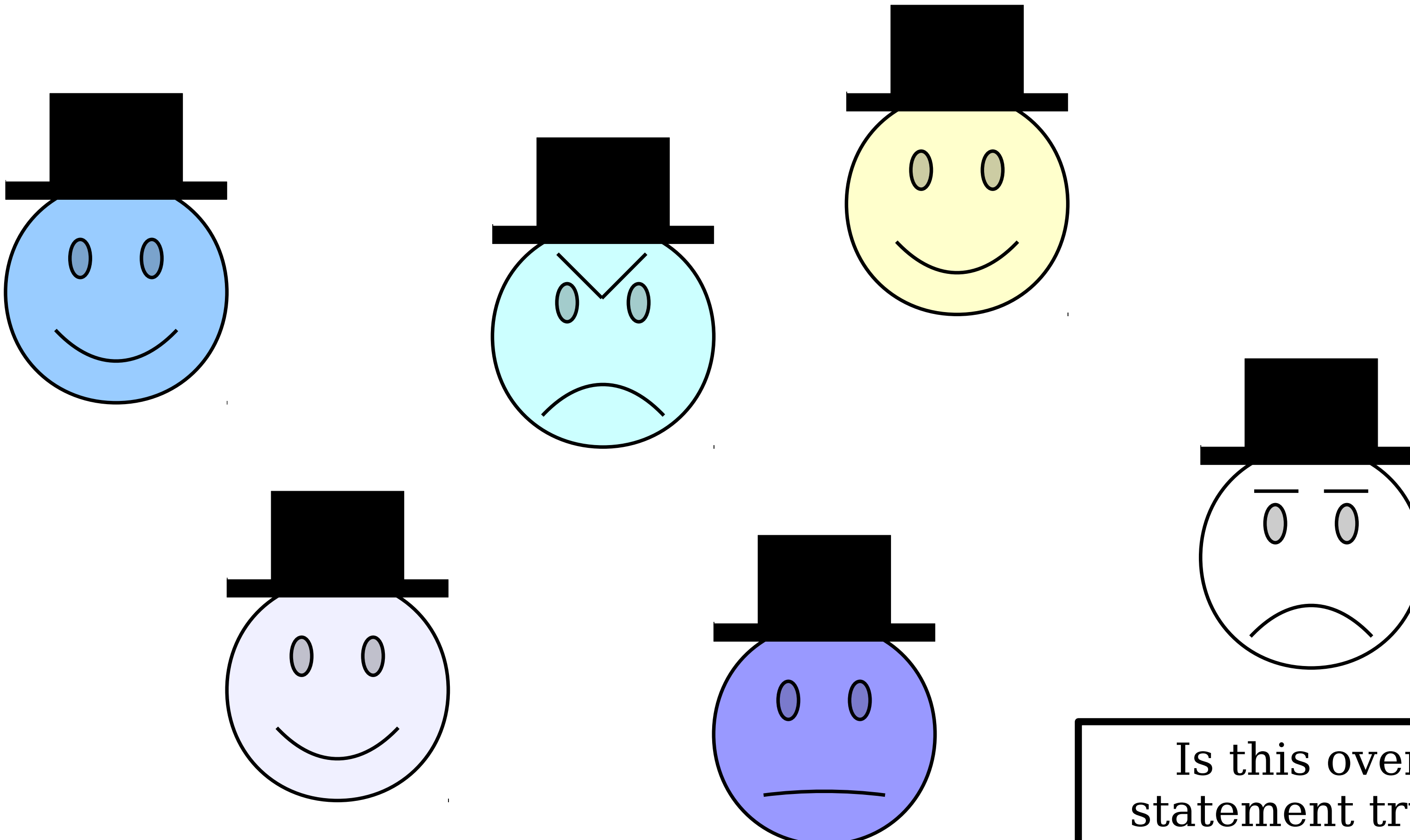
The Universal Quantifier



Is this part of the statement true or false?

~~$(\forall x.\ Smiling(x))$~~ $\to (\forall y.\ WearingHat(y))$

The Universal Quantifier



Is this overall statement true or false in this scenario?

$$(\forall x.\ Smiling(x)) \to (\forall y.\ WearingHat(y))$$

Fun with Edge Cases

Universally-quantified statements are said to be ***vacuously true*** in empty worlds.

$$\forall x.\ Smiling(x)$$

Translating into First-Order Logic

Translating Into Logic

- First-order logic is an excellent tool for manipulating definitions and theorems to learn more about them.
- Need to take a negation? Translate your statement into FOL, negate it, then translate it back.
- Want to prove something by contrapositive? Translate your implication into FOL, take the contrapositive, then translate it back.

Translating Into Logic

- When translating from English into first-order logic, we recommend that you

  think of first-order logic as a mathematical programming language.

- Your goal is to learn how to combine basic concepts (quantifiers, connectives, etc.) together in ways that say what you mean.

Using the predicates

- *Smiling*(*x*), which states that *x* is smiling, and
- *WearingHat*(*x*), which states that *x* is wearing a hat,

write a formula in first-order logic that says

some smiling person wears a hat.

How would you represent this in first-order logic?

Answer at
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Using the predicates

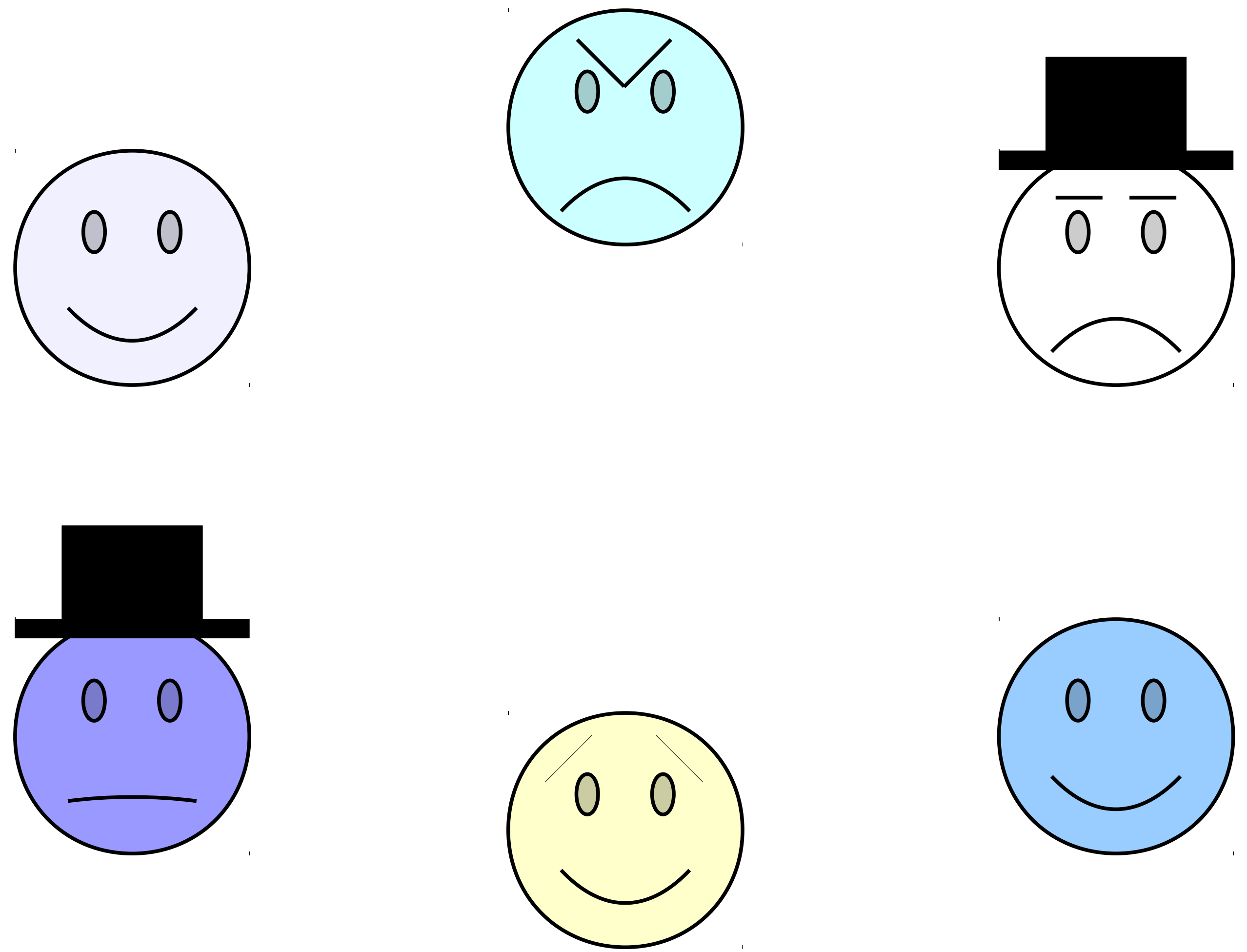
- $Smiling(x)$, which states that x is smiling, and
- $WearingHat(x)$, which states that x is wearing a hat,

write a formula in first-order logic that says

some smiling person wears a hat.

Which of the following are correct translations?

(A) ~~$\exists x.\ Smiling(Person(x))$~~
(B) ~~$\exists x.\ (Smiling(x) = WearingHat(x))$~~
(C) $\exists x.\ (Smiling(x) \land WearingHat(x))$
(D) $\exists x.\ (Smiling(x) \to WearingHat(x))$



"Some smiling person wears a hat."	***False***
$\exists x. (Smiling(x) \land WearingHat(x))$	***False***
~~$\exists x. (Smiling(x) \to WearingHat(x))$~~	***True***

“Some P is a Q”

translates as

$$\exists x. \, (P(x) \land Q(x))$$

Useful Intuition:

Existentially-quantified statements are false unless there's a positive example.

$$\exists x.\ (P(x) \land Q(x))$$

If x is an example, it *must* have property P on top of property Q.

Using the predicates

- *Smiling*(x), which states that x is smiling, and
- *WearingHat*(x), which states that x is wearing a hat,

write a sentence in first-order logic that says

every smiling person wears a hat.

Which of the following are correct translations?

(A) $\forall x.\ (Smiling(x) \land WearingHat(x))$

(B) $\forall x.\ (Smiling(x) \to WearingHat(x))$

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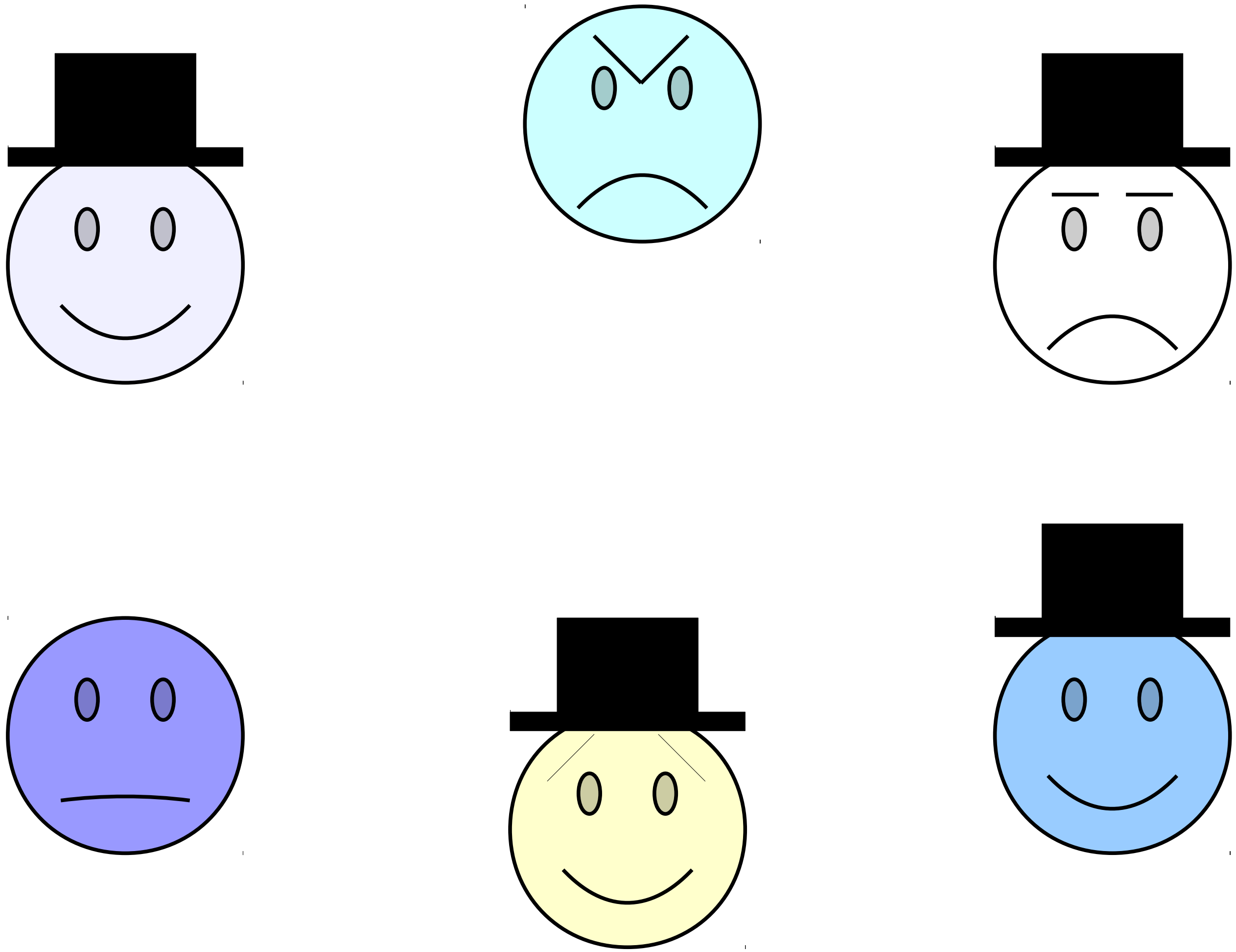

"Every smiling person wears a hat."	***True***
~~$\forall x.\ (Smiling(x) \land WearingHat(x))$~~	***False***
$\forall x.\ (Smiling(x) \to WearingHat(x))$	***True***

“All *P*'s are *Q*'s”

translates as

$$\forall x.\ (P(x) \to Q(x))$$

Useful Intuition:

Universally-quantified statements are true unless there's a counterexample.

$$\forall x.\ (P(x) \to Q(x))$$

If x is a counterexample, it *must* have property P but not have property Q.

Good Pairings

- The ∀ quantifier *usually* is paired with →.

$$\forall x.\ (P(x) \to Q(x))$$

- The ∃ quantifier *usually* is paired with ∧.

$$\exists x.\ (P(x) \land Q(x))$$

- In the case of ∀, the → connective prevents the statement from being *false* when speaking about some object you don't care about.
- In the case of ∃, the ∧ connective prevents the statement from being *true* when speaking about some object you don't care about.

Next Time

- ***First-Order Translations***
  - How do we translate from English into first-order logic?
- ***Quantifier Orderings***
  - How do you select the order of quantifiers in first-order logic formulas?
- ***Negating Formulas***
  - How do you mechanically determine the negation of a first-order formula?
- ***Expressing Uniqueness***
  - How do we say there’s just one object of a certain type?